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Asymptotics of Decay of Correlations in the ANNNI Model at High Temperatures

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The asymptotics of decay of correlations of spins in the ANNNI model on an $(\nu+1)$-dimensional lattice for high temperatures is shown to consist of exponentionally decaying oscillations. The problem of describing the asymptotics is reduced to the spectral analysis of one-particle states of a corresponding infinite-particle Hamiltonian.

KEY WORDS:

1. We consider the ANNNI model[1,2] on the lattice $\mathbb{Z}^\nu \times \mathbb{Z}^1$, which is defined by the Hamiltonian:

$$H = \sum_{t \in \mathbb{Z}^1} \sum_{x, x' \in \mathbb{Z}^\nu, |x-x'|=1} \sigma_{x,t}\sigma_{x',t} + J_1 \sum_{\substack{t,t' \in \mathbb{Z}^1 \\ |t-t'|=1}} \sum_{x \in \mathbb{Z}^\nu} \sigma_{x,t}\sigma_{x,t'} + J_2 \sum_{\substack{t,t' \in \mathbb{Z}^1 \\ |t-t'|=2}} \sum_{x \in \mathbb{Z}^\nu} \sigma_{x,t}\sigma_{x,t'} \tag{1}$$

where $\sigma_{x,t} = \pm 1$, $(x,t) \in \mathbb{Z}^{\nu+1}$, $J_1 > 0$, and $J_2 < 0$. We denote by $\sigma = \{\sigma_{x,t}, (x,t) \in \mathbb{Z}^{\nu+1}\}$ the Gibbs field defined by the Hamiltonian (1) (for small $\beta = 1/T$). Let $\sigma_n = \{\sigma_{x,n}\}$ be the values of configuration of the field on the layer $Y_n = \{(x,n)\}$.

In this paper we investigate the asymptotic behavior of the correlations

$$\langle F, F^{(n)} \rangle, \qquad n \to \infty \tag{2}$$

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where $F = F(\sigma_0) = \sigma_{0,0} + F_0(\sigma_0)$, $F_0(\sigma_0)$ is a "small" local functional on the values of the field on the zero layer Y_0, and $F^{(n)}$ is the same functional on the values of the field configuration on the nth layer Y_n (i.e., $F^{(n)}(\sigma_n) = F(\tau_{-n}\sigma_n)$, where τ_{-n} is a shift of the configuration $\sigma_{x,t}$ in the "time" direction on $-n$), and prove for $n \to \infty$ the following asymptotic formula:

$$\langle F, F^{(n)} \rangle = (-1)^{n+1} \frac{r_0^n}{n^{\nu/2}} \sin(n\tilde{\varphi}_0)[c + o(1)]$$

where $r_0 > 0$, $r_0 \sim \beta$, $\tilde{\varphi}_0 \sim \sqrt{\beta}$, and c is a constant. In particular, the asymptotics of decay of the correlations of spin $\langle \sigma_{0,0}, \sigma_{0,n} \rangle$ is of that kind.

To find the asymptotics of correlations (2), we use the scheme of refs. 3–5. According to refs. 4 and 5, we can rectify some low-degree (relative to the order of the parameter β) subspaces of the random field transfer matrix (since the field in the ANNNI model is a two-step Markov process, the constructions from of refs. 3–5 are slightly modified). Further, the spectrum of the transfer matrix in each of those subspaces is equivalent to the spectra of one-, two-, three- particle, etc., operators similar to the lattice Schrödinger operator. As a rule for the asymptotics (2) it is enough to know the one-particle spectrum. In the more general case (for example, for even functionals F on spin), one should use the structure of the spectrum of the transfer matrix in two-particle subspace, as was done in ref. 3. Note that the complete description of the two-particle spectrum was presented in ref. 6.

2. If we join neighbor layers Y_n of the lattice $\mathbb{Z}^{\nu+1}$ in one layer $\tilde{Y}_n = Y_{2n} \cup Y_{2n+1}$, we can construct, equivalent to the field σ, the Markov field η on the enlarged lattice $\bigcup_n \tilde{Y}_n = \mathbb{Z}^{\nu+1}$:

$$\eta = \{\eta_{x,t},\ (x, t) \in \mathbb{Z}^{\nu+1}\}$$

The spin value space S of this field consists of the ordered pairs $\{\eta^{(1)}, \eta^{(2)}\}$, $\eta^{(1)}, \eta^{(2)} = \pm 1$, and the Hamiltonian (1) takes the form

$$\begin{aligned} H = &\sum_{t \in \mathbb{Z}^1} \sum_{x, x' \in \mathbb{Z}^\nu, |x - x'| = 1} (\eta^{(1)}_{x,t}\eta^{(1)}_{x',t} + \eta^{(2)}_{x,t}\eta^{(2)}_{x',t}) \\ &+ J_1 \sum_{t \in \mathbb{Z}^1} \sum_{x \in \mathbb{Z}^\nu} (\eta^{(1)}_{x,t}\eta^{(2)}_{x,t} + \eta^{(1)}_{x,t}\eta^{(2)}_{x,t-1}) \\ &+ J_2 \sum_{t \in \mathbb{Z}^1} \sum_{x \in \mathbb{Z}^\nu} (\eta^{(1)}_{x,t}\eta^{(1)}_{x,t-1} + \eta^{(2)}_{x,t}\eta^{(2)}_{x,t-1}) \end{aligned} \tag{3}$$

This Hamiltonian is not invariant to "inversion of time": $(x, t) \to (x, -t)$, and therefore the transfer matrix $\mathcal{F}$ of the field η

$$\mathcal{F}F = P_{\mathcal{H}_{\text{ph}}}\mathcal{U}_1 F, \qquad F \in \mathcal{H}_{\text{ph}}$$

is not a self-adjoint operator in $\mathcal{H}_{\text{ph}}$. The operator $\mathcal{F}^*$ adjoint to $\mathcal{F}$ acts according to

$$\mathcal{F}^*F = P_{\mathcal{H}_{\text{ph}}}\mathcal{U}_{-1} F, \qquad F \in \mathcal{H}_{\text{ph}}$$

Here $\mathcal{H}_{\text{ph}} = L_2^{\tilde{Y}_0} \subset L_2(S^{\mathbb{Z}^{\nu+1}}, \mu)$ is the Hilbert space of the functionals, dependent on values of the field η on the zero layer $\tilde{Y}_0$; $P_{\mathcal{H}_{\text{ph}}}$ is the orthogonal projection in $L_2(S^{\mathbb{Z}^{\nu+1}}, \mu)$ on $\mathcal{H}_{\text{ph}}$; $\mathcal{U}_{\pm 1}$ are shift operators in $L_2(S^{\mathbb{Z}^{\nu+1}}, \mu)$ in the time direction (forward and backward); μ is the limit Gibbs measure on the space $S^{\mathbb{Z}^{\nu+1}}$, generated by the Hamiltonian (3).

Using the methods developed in refs. 4 and 5, it is possible to prove the following theorem.

Theorem. For β small, $0 < \beta < \beta_0$ there exist two decompositions of $\mathcal{H}_{\text{ph}}$ in a direct (nonorthogonal, in general) sum of subspaces, invariant to operators $\mathcal{F}$ and $\mathcal{F}^*$, respectively, and a group of space shifts $\{\mathcal{U}_x, x \in \mathbb{Z}^\nu\}$:

$$\mathcal{H}_{\text{ph}} = \mathcal{H}_1 + \mathcal{H}_2 = \mathcal{H}_1^* + \mathcal{H}_2^* \tag{4}$$

In this case

$$\mathcal{H}_2 = (\mathcal{H}_1^*)^\perp, \qquad \mathcal{H}_2^* = (\mathcal{H}_1)^\perp \tag{5}$$

and the norms of the operators $\mathcal{F}$ and $\mathcal{F}^*$ on the subspaces $\mathcal{H}_2$ and $\mathcal{H}_2^*$, respectively, are of order β^2, i.e.,

$$\|\mathcal{F}\,|_{\mathcal{H}_2}\| \sim \beta^2, \qquad \|\mathcal{F}^*\,|_{\mathcal{H}_2^*}\| \sim \beta^2 \tag{6}$$

There exist two biorthogonal bases $\{h_x^{(j)}, x \in \mathbb{Z}^\nu, j = 1, 2\}$ and $\{\hat{h}_x^{(j)}, x \in \mathbb{Z}^\nu, j = 1, 2\}$ in the subspaces $\mathcal{H}_1$ and $\mathcal{H}_1^*$, respectively, such that $(h_x^{(j)}, \hat{h}_{x'}^{(k)})_{\mathcal{H}_{\text{ph}}} = \delta_{x,x'}\delta_{j,k}$,

$$\mathcal{U}_s h_x^{(j)} = h_{x+s}^{(j)}, \qquad \mathcal{U}_s \hat{h}_x^{(j)} = \hat{h}_{x+s}^{(j)}, \qquad x, s \in \mathbb{Z}^\nu, \qquad j = 1, 2$$

and

$$\begin{aligned} \mathcal{F}\,|_{\mathcal{H}_1} h_x^{(j)} &= \sum_{k=1,2;\, x' \in \mathbb{Z}^\nu} a_{x-x'}^{j,k} h_{x'}^{(k)} \\ \mathcal{F}^*\,|_{\mathcal{H}_1^*} \hat{h}_x^{(j)} &= \sum_{k=1,2;\, x' \in \mathbb{Z}^\nu} \hat{a}_{x-x'}^{j,k} \hat{h}_{x'}^{(k)} \end{aligned} \tag{7}$$

Here $\hat{A}_x = A_x^*$, $A_x = \|a_x^{j,k}\|$, and $\hat{A}_x = \|\hat{a}_x^{j,k}\|$ are second-order matrices, and A_x has the decomposition

$$\begin{aligned} A_0 &= \beta \begin{pmatrix} J_2 & J_1 \\ 0 & J_2 \end{pmatrix} + \beta^2 \begin{pmatrix} \frac{1}{2}J_1^2 & J_1 J_2 \\ J_1 J_2 & \frac{1}{2}J_1^2 \end{pmatrix} + O(\beta^3) \\ A_x &= \beta^2 \begin{pmatrix} J_2 & J_1 \\ 0 & J_2 \end{pmatrix} + O(\beta^3) \qquad \text{for} \quad |x| = 1 \end{aligned} \tag{8}$$

$$|a_x^{j,k}| \leqslant c\beta^{|x|+1} \qquad \text{for} \quad |x| \geqslant 2, \quad c \text{ is a constant}$$

Remark. Both of the biorthogonal bases $\{h_x^{(j)}, x \in \mathbb{Z}^\nu, j = 1, 2\}$ and $\{\hat{h}_x^{(j)}, x \in \mathbb{Z}^\nu, j = 1, 2\}$ on subspaces $\mathscr{H}_1$ and $\mathscr{H}_1^*$ are constructed as perturbations of a system of functions $\{\eta_{x,0}^{(j)}, x \in \mathbb{Z}^\nu, j = 1, 2\}$, and we have

$$h_x^{(j)} = \eta_{x,0}^{(j)} - \beta \sum_{y<x} \eta_{y,0}^{(j)} - \tfrac{1}{2} J_1 \beta \eta_{x,0}^{(3-j)} + g_x^{(j)}$$

$$\hat{h}_x^{(j)} = \eta_{x,0}^{(j)} - \beta \sum_{y<x} \eta_{y,0}^{(j)} - \tfrac{1}{2} J_1 \beta \eta_{x,0}^{(3-j)} + \hat{g}_x^{(j)}$$

where $g_x^{(j)}$, $\hat{g}_x^{(j)}$, $j = 1, 2$, are functions of order of β^2, x, $y \in \mathbb{Z}^\nu$, $y < x$ in lexicographic order.

3. Let $F \in \mathscr{H}_{\text{ph}}$ be a functional such that

$$F = \sum_{(x,j) \in B \subset \tilde{Y}_0} k_x^{(j)} \eta_{x,0}^{(j)} + \sum_{I \subset \tilde{Y}_0}{}' k_I \eta_I \tag{9}$$

where B and I are some finite subsets of the layer $\tilde{Y}_0$, $\sum'$ means a sum over a finite number of subsets $I \subset \tilde{Y}_0$, $\eta_I = \prod_{(x,j) \in I} \eta_{x,0}^{(j)}$, and $k_x^{(j)} \neq 0$ and $k_I \neq 0$ are real constants.

Lemma. For a functional $F \in \mathscr{H}_{\text{ph}}$ of the form (9) in the decomposition $F = F_1 + F_2 = F_1^* + F_2^*$, where

$$F_1 = \sum_{x \in \mathbb{Z}^\nu, j=1,2} c_x^{(j)} h_x^{(j)}, \qquad F_1^* = \sum_{x \in \mathbb{Z}^\nu, j=1,2} d_x^{(j)} \hat{h}_x^{(j)}$$

the coefficients $c_x^{(j)}$ and $d_x^{(j)}$ are real and have the upper bounds

$$|c_x^{(j)}| < c_1 \lambda^{|x|}, \qquad |d_x^{(j)}| < c_2 \lambda^{|x|}$$

for some $0 < \lambda < 1$, and

$$\sum_{x \in \mathbb{Z}^\nu, j=1,2} |c_x^{(j)}| > M, \qquad \sum_{x \in \mathbb{Z}^\nu, j=1,2} |d_x^{(j)}| > M$$

where M is a constant depending on constants $\{k_x^{(j)}, x \in B, j=1, 2\}$, and c_1 and c_2 are constants depending on B. Moreover, for the norms of $F_2 \in \mathscr{H}_2$ and $F_2^* \in \mathscr{H}_2^*$ the following hold:

$$\|F_2\| < c \, \|F\|, \qquad \|F_2^*\| < c \, \|F\|$$

with some absolute constant c.

Now using (4)–(6), we have

$$\begin{aligned}\langle F, F^{(n)} \rangle &= (\mathscr{F}^n F, \overline{F}) \\ &= (\mathscr{F}^n (F_1 + F_2), F_1^* + F_2^*) \\ &= (\mathscr{F}^n F_1, F_1^*) + (\mathscr{F}^n F_2, F_2^*) = (\mathscr{F}_1^n F_1, F_1^*) + O(\beta^{2n}) \quad (10)\end{aligned}$$

where $\mathscr{F}_1 = \mathscr{F} \,|_{\mathscr{H}_1}$.

Hence the main term of the asymptotics of the correlations (2) is determined by the expression $(\mathscr{F}_1^n F_1, F_1^*)$. The lemma and the properties of biorthogonal bases imply

$$(\mathscr{F}_1^n F_1, F_1^*) = \sum_{x, x' \in \mathbb{Z}^\nu; j, k=1, 2} c_x^{(j)} d_{x'}^{(k)} b_{x-x'}^{(n), jk} \qquad (11)$$

where $B_{x-x'}^{(n)} = \|b_{x-x'}^{(n), jk}\|$ is the n-repeated convolution of the matrix $A_{x-x'}$. Performing a Fourier transform for the variable x,

$$h_x^{(j)} \to e^{i(\lambda, x)}, \qquad \hat{h}_x^{(j)} \to e^{i(\lambda, x)}$$

$$\sum_{x \in \mathbb{Z}^\nu} c_x^{(j)} h_x^{(j)} \to \sum_{x \in \mathbb{Z}^\nu} c_x^{(j)} e^{i(\lambda, x)} = f_j(\lambda) \in L_2(T^\nu)$$

$$\sum_{x \in \mathbb{Z}^\nu} d_x^{(j)} \hat{h}_x^{(j)} \to \sum_{x \in \mathbb{Z}^\nu} d_x^{(j)} e^{i(\lambda, x)} = \hat{f}_j(\lambda) \in L_2(T^\nu)$$

for every $j = 1, 2$, we obtain the following expression for the bilinear form (11):

$$\begin{aligned}&\int_{T^\nu} (B^n(\lambda) \, f(\lambda), \hat{f}(\lambda)) \, d\lambda \\ &\quad = \sum_{j,k=1,2} \int_{T^\nu} B_{j,k}^{(n)}(\lambda) \, f_j(\lambda) \, \overline{\hat{f}_k(\lambda)} \, d\lambda \qquad (12)\end{aligned}$$

where the matrix $B(\lambda)$ is a Fourier transform of the series of matrices $\{A_x, x \in \mathbb{Z}^\nu\}$, $B^n(\lambda) = \{B_{j,k}^{(n)}(\lambda)\}$, and $B(\lambda)$ satisfies the decomposition

$$B(\lambda) = A_0 + 2A_{|x|=1} \sum_{j=1}^{\nu} \cos \lambda^{(j)} + O(\beta^3)$$

where A_0 and $A_{|x|=1}$ are determined in (8).

Remark. It can be seen from (7) that the spectrum of the operator $\mathscr{F}_1$ coincides with the spectrum of the operator $(Bf)(\lambda) = B(\lambda) f(\lambda)$, which acts in the space $L_2(T^\nu, C^2)$ of vector-valued functions on a ν-dimensional torus.

The decomposition of the vector $f(\lambda)$ by eigenfunctions of the operator $B(\lambda)$ is

$$f(\lambda) = h_1(\lambda) e_1(\lambda) + h_2(\lambda) e_2(\lambda) \tag{13}$$

Since the eigenvalues of the matrix $B(\lambda)$,

$$\omega_{1,2}(\lambda) = J_2\beta \pm iJ_1 |J_2|^{1/2} \beta^{3/2} + \tfrac{1}{2} J_1^2 \beta^2 + 2J_2\beta^2 \sum_{j=1}^{\nu} \cos\lambda^{(j)} + O(\beta^{5/2}) \tag{14}$$

are different for every $\lambda \in T^\nu$: $\omega_1(\lambda) \neq \omega_2(\lambda)$, $\lambda \in T^\nu$, the functions $h_1(\lambda)$ and $h_2(\lambda)$ from (13) are smooth. Using an analogous decomposition of $\hat{f}(\lambda)$ by eigenfunctions of the operator $B^*(\lambda)$ from (12) and (13) we have

$$\begin{aligned}
&\int_\nu (B^n(\lambda) f(\lambda), \hat{f}(\lambda))\, d\lambda \\
&= \int_{T^\nu} (\omega_1^n(\lambda) h_1(\lambda) e_1(\lambda) \\
&\quad + \omega_2^n(\lambda) h_2(\lambda) e_2(\lambda),\, g_1(\lambda) \hat{e}_1(\lambda) + g_2(\lambda) \hat{e}_2(\lambda))\, d\lambda \\
&= \int_{T^\nu} \omega_1^n(\lambda) \psi_1(\lambda)\, d\lambda + \int_{T^\nu} \omega_2^n(\lambda) \psi_2(\lambda)\, d\lambda
\end{aligned} \tag{15}$$

The functions $\omega_{1,2}(\lambda)$ are even; thus, there exists only the critical point $\lambda_0 = 0$ on T^ν for both functions $\omega_1(\lambda)$ and $\omega_2(\lambda)$ such that the absolute values of the functions in it have their absolute maximum: $|\omega_1(\lambda_0)| = |\omega_2(\lambda_0)| = r_0$, and $\arg \omega_1(\lambda_0) = -\arg \omega_2(\lambda_0) = \varphi_0$, and also

$$\overline{\psi_1(\lambda_0)} = \psi_2(\lambda_0) \tag{16}$$

From (10)–(12) and (14)–(16), using the method of stationary phase for every integral in (15), we have

$$\begin{aligned}
(\mathscr{F}^n F, F) &= \frac{r_0^n}{n^{\nu/2}} \cos(n\varphi_0 + \alpha)[c + o(1)] \\
&= (-1)^{n+1} \frac{r_0^n}{n^{\nu/2}} \sin(n\tilde{\varphi}_0)[c + o(1)]
\end{aligned}$$

Here

$$r_0 = |J_2|\ \beta + 2v\ |J_2|\ \beta^2 + O(\beta^{5/2})$$

$$\varphi_0 = \pi - \frac{J_1}{|J_2|^{1/2}} \beta^{1/2} + O(\beta)$$

$$\alpha = \arg \psi_1(\lambda_0) + \arg[(-(\ln \omega_1(\lambda))''|_{\lambda=\lambda_0})^{-1/2}]$$

$$= -\frac{\pi}{2} + O(\beta^{1/2})$$

$$\tilde{\varphi}_0 = p\beta^{1/2} + O(\beta)$$

where p and c are constants.

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